Exact Solution for Buckling of Columns Including Self-Weight

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Abstract: In this technical note, analytical solutions for the elastic buckling of heavy columns with various combinations of end conditions are derived, for the first time, in terms of generalized hypergeometric functions. The benchmark solutions may be used to assess the accuracy of approximate formulas and numerical solutions.


CE Database subject headings: Columns; Buckling; Numerical models; Axial loads; Weight.

Introduction

In conventional column buckling problems, the self-weight is often neglected since it is assumed to be small when compared to the applied axial loads. So when the column’s self-weight is relatively significant and therefore it has to be taken into consideration in the buckling analysis, the column is generally referred to as a “heavy” column. The elastic buckling of a heavy column (i.e., column buckling under its own weight) was first investigated by Greenhill (1881). He found that the maximum length of a heavy cantilevered column is \( L_{\text{max}} = \sqrt{7.8375p/D} \), where \( p \) = weight per unit length and \( D \) = flexural rigidity of the column. This length, interestingly, sets the maximum height for trees if we assume the trees to be prismatic and the branches are neglected. About 5 decades later, Grishcoff (1930) obtained the buckling load for the combined effect of both the self-weight and an axial load for a cantilevered column by using an infinite series. Wang (1983) and Wang and Drachman (1981) extended the work to include the case of a finite column hanging from its foundation. Analytical solutions are obtained for the cases of fixed ends or pinned ends. However, their exact solutions are derived from a second-order differential equation which blurs the difference between the cases involving a fixed end and a sliding restraint and that of a pinned end and a free end. In the Handbook of Structural Stability (Kato 1971) produced by the Column Research Committee of Japan, one may find Kato’s approximate formulas for the buckling loads of heavy columns with various end conditions. Using the Rayleigh–Ritz method and the appropriate choice of trigonometric functions, Wang and Ang (1988) derived approximate buckling formulas that are simpler and furnish more accurate buckling solutions when compared to those obtained by Kato.

Obviously, it is better to obtain exact solutions wherever possible as approximate formulas can only furnish accurate solutions for a limited range of input parameters. Noting the difficulty in obtaining exact solutions, Chai and Wang (2006) proposed the use of the differential transformation method for solving the governing differential equation of the heavy column buckling problem. This method was first proposed by Pukhov (Abbasov and Bahadir 2005) in the 1970s for solving electronics problems. It is a powerful method because it could convert differential equations into a set of recursive algebraic equations without the need for integration. However, quite a fairly large number of recursive terms are required for accurate solutions.

By complementing previous researchers work on this interesting heavy column buckling problem, we present the analytical solutions in terms of a generalized hypergeometric function. The generalized hypergeometric functions have been typically used when studying the dynamics of inhomogeneous plates and beams (Duan et al. 2005; Elishakoff 2005; Wang 1967). The proposed analytical solutions are obtained for various combinations of end conditions, giving the most up-to-date set of analytical solutions for the aforementioned buckling problem.

Problem Definition and Formulation

Fig. 1 shows a column of length \( L \), flexural rigidity \( D \), subjected to both its own weight \( p \) per unit length and an end concentrated load \( P \). The \( x-y \) axes are also shown in the figure. The ends of the column may be clamped, pinned, free, or have a sliding restraint. The problem at hand is to determine the buckling capacity of such a loaded column.

The governing differential equation for the buckling of such a loaded column is given by (Chai and Wang 2006)

\[
d^4w \over dx^4 + [\alpha + \beta(1-x)] \frac{d^2w}{dx^2} - \beta \frac{dw}{dx} = 0
\]

where
Fig. 1. Heavy column under axial load: (a) \( \alpha > 0, \beta > 0 \); (b) \( \alpha > 0, \beta < 0 \)

\[ \alpha = \frac{PL^2}{D}; \quad \beta = \frac{PL^3}{D}; \quad x = \frac{\bar{x}}{L}; \quad w = \frac{\bar{w}}{L} \] \hspace{1cm} (2)

= nondimensional parameters representing the end load, self-weight, coordinates, and transverse deflection, respectively. \( \alpha \) and \( \beta \) are both positive as shown in Fig. 1(a). Negative \( \alpha \) means that the weight and end load act in opposite directions. Negative \( \beta \) can be realized in the case of a hanging column or a marine riser as shown in Fig. 1(b).

The various boundary conditions considered are given by (Wang et al. 2005):

1. Clamped end (C)
   \[ w = 0, \quad \frac{dw}{dx} = 0 \]

2. Pinned end (P)
   \[ w = 0, \quad \frac{d^2w}{dx^2} = 0 \]

3. Free end (F)
   \[ \frac{d^2w}{dx^2} = 0, \quad \frac{d^3w}{dx^3} + \frac{d}{dx} \frac{dw}{dx} = 0 \]

4. Sliding restraint (S)
   \[ \frac{dw}{dx} = 0, \quad \frac{d^2w}{dx^2} + \frac{d}{dx} \frac{dw}{dx} = 0 \]

Therefore the combination of these boundary condition results in five cases, i.e., C-F, P-P, P-C, C-S, and C-C columns.

**Generalized Hypergeometric Function Solution**

By letting
\[ x = 1 + \frac{\alpha}{\beta} + \left( \frac{9r}{\beta} \right)^{1/3} \] \hspace{1cm} (3)

Eq. (1) can be converted to
\[ r^2 \frac{d^2w}{dr^2} + 4r \frac{d^3w}{dr^3} + \left( \frac{20}{9} - r \right) \frac{d^2w}{dr^2} - \frac{dw}{dr} = 0 \] \hspace{1cm} (4)

Eq. (4) is the generalized hypergeometric equation and can be expressed as follows:

\[ \partial \left( \partial \frac{1}{\partial} \left( \partial \frac{1}{\partial} \right) \right) w(r) = r \partial^3 w(r) \] \hspace{1cm} (5)

where \( \partial = \partial \partial / \partial r \) = differential operator.

The boundary conditions may be expressed in terms of the variable \( r \) as

1. Clamped end (C)
   \[ w = 0, \quad r^{2/3} \frac{dw}{dr} = 0 \] \hspace{1cm} (6)

2. Pinned end (P)
   \[ w = 0, \quad r^{1/3} \left( \frac{d^2w}{dr^2} + 3r \frac{d^2w}{dr^3} \right) = 0 \] \hspace{1cm} (7)

3. Free end (F)
   \[ r^{1/3} \left( \frac{d^2w}{dr^2} + 3r \frac{d^2w}{dr^3} \right) = 0 \]

4. Sliding restraint (S)
   \[ r^{2/3} \frac{dw}{dr} = 0 \]

\[ \frac{1}{3} \beta \left( \frac{d^2w}{dr^2} + 18r \frac{d^2w}{dr^3} + 9r^2 \frac{d^2w}{dr^3} \right) + \alpha(3\beta)^{1/3} r^{2/3} \frac{dw}{dr} = 0 \] \hspace{1cm} (8)

Eq. (5) is equivalent to

\[ \left[ \sum_{j=1}^{q} \frac{1}{\partial} \left( \frac{1}{\partial} \left( \frac{1}{\partial} \right) \right) w(r) = 0 \right] \]

The coefficients involved in Eq. (10) are calculated from

\[ q = 3; \quad p = 2; \quad a_1 = 0; \quad a_2 = 0; \quad a_3 = 0; \quad b_1 = \frac{2}{3}; \quad b_2 = \frac{1}{3}; \quad b_3 = 0 \] \hspace{1cm} (11)

Since no two \( b_i \) are equal or differ by an integer, the solution of Eq. (10) can be obtained (Rainville 1960; Zwillinger 1992)

\[ w(r) = \sum_{m=0}^{3} c_m w_m(r) \] \hspace{1cm} (12)

with

\[ w_0(r) = F(a_1, a_2, b_1, b_2, b_3; r) \]

\[ w_1(r) = r^{1-b_1} F(a_1 - b_1 + 1, a_2 - b_1 + 1; 2 - b_1, b_2 - b_1 + 1; 1) \]

\[ w_2(r) = r^{1-b_2} F(a_1 - b_2 + 1, a_2 - b_2 + 1; b_1 - b_2 + 1, 2 - b_2 + 1; r) \]

\[ w_3(r) = r^{1-b_3} F(a_1 - b_3 + 1, a_2 - b_3 + 1; b_1 - b_3 + 1, 2 - b_3 + 1; r) \]

where the generalized hypergeometric function

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Problem based on these conditions can be solved numerically

The second-order differential equation which blurred their differ-

ences because the boundary conditions are not being captured (Wang et al. 2005). The buckling capacities for the C-S column are new and they have not been hitherto reported in the literature.

In Fig. 2, the α intercepts the curves for C-F, P-P, C-S, C-P, and C-C end supports are 2.4664, 9.8688, 9.8691, 20.1904, and 39.4782, respectively, which agree with the normalized Euler buckling loads $\pi^2/4$, $\pi^2$, $\pi^2$, $2.0457\pi^2$, $4\pi^2$ given in Wang et al. (2005).

The critical value of $\beta$ for which $\alpha=0$ is important from a designer’s point of view because it furnishes the maximum feasible column length. A set of formulas for $\beta$ under $\alpha=0$ for various end supports are obtained as follows:

1. C-F

$$\beta = \frac{f_1}{8}$$

2. P-P

$$\frac{f_5}{f_6} = \frac{8\beta^2 f_5 - 280\beta f_2 - 3360 f_1}{3\beta^2 f_6 - 336 f_2}$$

3. C-P

$$\frac{f_3}{f_6} = \frac{5f_3 + 80}{3f_6}$$

4. C-C

$$\frac{f_4}{f_6} = \frac{4f_4 + 40}{3f_6}$$

5. C-S

$$\frac{\beta}{20} = \frac{f_2}{f_6}$$

where

$$f_1 = iF_2\left(\frac{1}{3}, \frac{4}{3}; 3, -\frac{\beta}{9}\right), \quad f_2 = iF_2\left(\frac{2}{3}, \frac{5}{3}; 3, -\frac{\beta}{9}\right),$$

$$f_3 = iF_2\left(\frac{5}{3}, \frac{7}{3}; 3, -\frac{\beta}{9}\right), \quad f_5 = iF_2\left(\frac{7}{3}, \frac{10}{3}; 3, -\frac{\beta}{9}\right),$$

$$f_4 = iF_2\left(\frac{1}{3}, \frac{4}{3}; 3, -\frac{\beta}{9}\right), \quad f_6 = iF_2\left(\frac{2}{3}, \frac{5}{3}; 3, -\frac{\beta}{9}\right),$$

$$f_7 = iF_2\left(\frac{5}{3}, \frac{7}{3}; 3, -\frac{\beta}{9}\right), \quad f_8 = iF_2\left(\frac{7}{3}, \frac{10}{3}; 3, -\frac{\beta}{9}\right).$$

The first critical values of $\beta$ for which $\alpha=0$ with various end supports are listed in Table 1 and these values may be used to

Results and Discussion

Based on the hypergeometric solution, numerical values of the buckling capacities for heavy columns are computed and presented in Fig. 2 for the various end support conditions. These curves can be viewed as stability criteria for the buckling of heavy columns. When the values of the self-weight parameter $\beta$ and the axial force parameter $\alpha$ are found under the curves, the column will not buckle.

It is evident from Fig. 2 that the buckling capacities curves are nonlinear and highly dependent on the end support conditions and the weight of the columns. The curves are different for the cases of columns with fixed end and sliding restraint and columns with pinned end and free end. Therefore it is necessary to use the fourth-order differential equation to distinguish them instead of the second-order differential equation which blurred their differ-

![Fig. 2. Buckling capacities of heavy columns](http://www.asclibrary.org)
assess the accuracy of Wang and Ang’s approximate formulas (Wang and Ang 1988) which are also listed in Table 1. It can be observed that Wang and Ang’s approximate formulas overpredict the buckling load parameter $\beta$ by up to 5.8% for $C-C$ and 2.7% for $C-P$ columns.

References


